CHAPTER 1 EM Quadratic Equations

1) Complete the square (Changing $y = ax^2 + bx + c$ to $y = a(x-h)^2 + K$)

$$x^2 + bx + c = \underbrace{x^2 + bx + \left(\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right)^2 + c$$
$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

"number in front of x^2"
* coefficient of x^2 must be positive 1 before you start completing the square.

Example 1 : When coefficient of x^2 is +1

	Steps	Example : $x^2 - 10x - 8$
1)	Copy the x^2 term and bx term. Add $\left(\frac{b}{2}\right)^2$, subtract $\left(\frac{b}{2}\right)^2$. Copy the constant.	$x^2 - 10x - 8$ $\boxed{b = -10}$ $= \underbrace{x^2 - 10x + \left(\frac{-10}{2}\right)^2} - \left(\frac{-10}{2}\right)^2 - 8$
2)	First three terms becomes $\left(x + \frac{b}{2}\right)^2$	$= \left(x - \frac{10}{2}\right)^2 \underbrace{- \left(\frac{-10}{2}\right)^2 - 8}$
3)	Combine the constants.	$= (x-5)^2 - 33$

Example 2 : When coefficient of x^2 is NOT +1.

	Steps	Example : $-x^2 + 5x + 7$
1)	Factorise the coefficient of x^2. Leave the constant alone.	$-x^2 + 5x + 7$ $= -(x^2 - 5x) + 7$ $\boxed{b = -5}$ complete the square for this bracket. Everything else outside the bracket stays the same.
2)	Copy the x^2 term and bx term. Add $\left(\frac{b}{2}\right)^2$, subtract $\left(\frac{b}{2}\right)^2$. Copy the constant.	$= -\left[\underbrace{x^2 - 5x + \left(\frac{-5}{2}\right)^2} - \left(\frac{-5}{2}\right)^2\right] + 7$
3)	First three terms becomes $\left(x + \frac{b}{2}\right)^2$	$= -\left[\left(x + \frac{-5}{2}\right)^2 - \left(\frac{-5}{2}\right)^2\right] + 7$
4)	Expand the outer bracket.	$= -\left(x + \frac{-5}{2}\right)^2 \underbrace{+ \left(\frac{-5}{2}\right)^2 + 7}$
5)	Combine the constants.	$= -\left(x - \frac{5}{2}\right)^2 + \frac{53}{4}$

2) Solving quadratic equations

" Find value of x "

Calculator Hack! Check answers using calculator by pressing: menu $\rightarrow$ 5 $\rightarrow$ 2 $\rightarrow$ 2

a) Factorisation (Frame / Table)

$\leftarrow$ Learnt in Sec 2

E.g. Solve $x^2 - 5x - 6 = 0$

$$(x-6)(x+1) = 0$$

$$x - 6 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 6 \qquad\qquad x = -1$$

Frame

x	x	-6
x	x^2	$-6x$
1	x	-6

Table

x	-6	$-6x$
x	1	x
x^2	-6	$-5x$

b) Completing the square

E.g. $x^2 - 6x + 5 = 0$

complete the square

$$(x-3)^2 - 4 = 0$$

$$(x-3)^2 = 4$$

square root & include $\pm$

$$x - 3 = \pm 2$$

$$x = 2 + 3 \quad \text{or} \quad x = -2 + 3$$

$$x = 5 \quad \text{or} \quad x = 1$$

E.g. $x^2 - 4x + 7 = 0$

complete the square

$$(x-2)^2 + 3 = 0$$

$$(x-2)^2 = -3$$

$\leftarrow$ We cannot square root negative numbers.

As $(x-2)^2 \geq 0$ for all real values of x, $(x-2)^2 = -3$ has no real solutions.

c) Quadratic formula

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Eg: Solve $-3x - 17x^2 = -3$

$$-17x^2 - 3x + 3 = 0$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-17)(3)}}{2(-17)}$$

$$= -0.517 \text{ or } 0.341 \text{ (3s.f.)}$$

Steps

① Rearrange into this form: $ax^2 + bx + c = 0$

② Identify a, b, c

③ Substitute the values into the quadratic formula

d) Graphical Method $\rightarrow$ E.g. Draw the graph of $y = 2x^2 - 5x + 4$

(i) Using your graph, solve $2x^2 - 5x + 4 = 0$

$\rightarrow$ Read the x-intercept(s) where the curve cuts the x-axis.

(ii) Solve $2x^2 + x - 7 = 0$

$$-6x \qquad\qquad -6x$$

$$2x^2 - 5x - 7 = -6x$$

$$+11 \qquad\qquad +11$$

$$2x^2 - 5x + 4 = -6x + 11$$

$$y = -6x + 11$$

$\rightarrow$ Draw this line in your graph

x			
y			

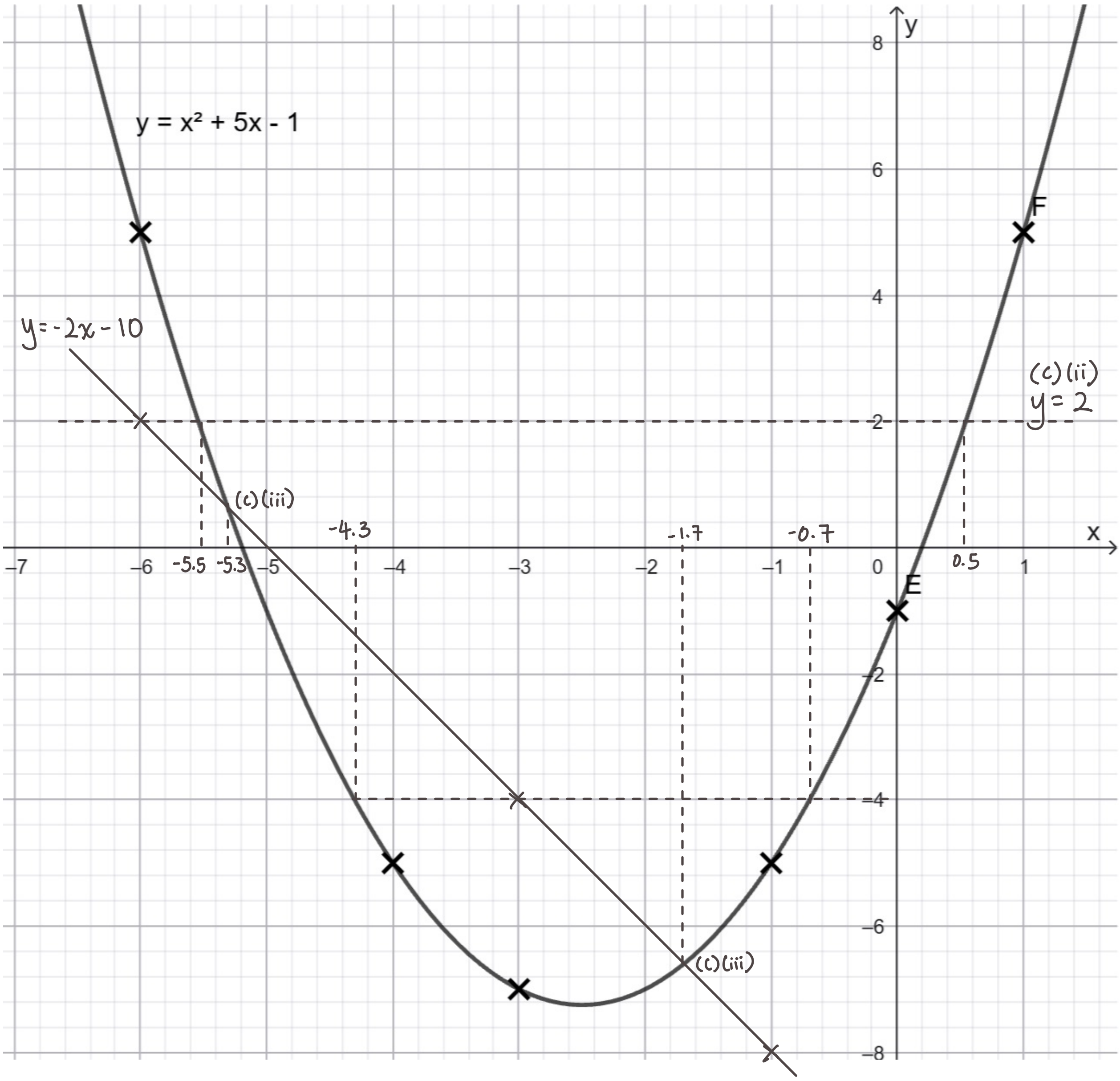
$\rightarrow$ Read the x-coordinates of the intersection points.

* Make the LHS of the given equation the same as the graph equation. Change term by term.

Eg: (a) Complete the table of values for $y = x^2 + 5x - 1$

x	-6	-4	-3	-1	0	1
y	5	-5	-7	-5	-1	5

(b) On the grid below, draw the graph of $y = x^2 + 5x - 1$ for $-6 \le x \le 1$



(c) Use your graph to

(i) write down an inequality in x to describe the range of values where $y < -4$.

$-4.3 < x < -0.7$

(ii) solve $x^2 + 5x - 3 = 0$

$x^2 + 5x - 1 = 2$

Draw $y = 2$

From graph, $x = -5.5$ or 0.5

(iii) solve $2x^2 + 7x + 9 = x^2$

$x^2 + 7x + 9 = 0$

$x^2 + 5x + 9 = -2x$

$x^2 + 5x - 1 = -2x - 10$

Draw $y = -2x - 10$

x	-6	-3	-1
y	2	-4	-8

From graph, $x = -5.3$ or -1.7

3) Solving fractional equations

$$\frac{6}{3-2x} + \frac{5x}{4x^2-9} = 3$$

TIP

Factorise -1 to change $3-2x$ to $2x-3$

Steps to solve

$$\frac{6}{3-2x} + \frac{5x}{(2x+3)(2x-3)} = 3$$

① Fully factorise the denominator
↳ Frame / Table / Special identities

$$-\frac{6}{2x-3} + \frac{5x}{(2x+3)(2x-3)} = 3$$

$$-\frac{6(2x+3)}{(2x+3)(2x-3)} + \frac{5x}{(2x+3)(2x-3)} = 3$$

② Make the denominators the same

$$-\frac{6(2x+3)+5x}{(2x+3)(2x-3)} = 3$$

③ Combine into one single fraction.

$$\frac{-12x-18+5x}{(2x+3)(2x-3)} = 3$$

④ Simplify numerator.

$$\frac{-7x-18}{(2x+3)(2x-3)} = 3$$

$$-7x-18 = 3(2x+3)(2x-3)$$

⑤ Cross multiply

$$-7x-18 = 3(4x^2-9)$$

⑥ Expand and simplify

$$-7x-18 = 12x^2-27$$

$$12x^2+7x-9 = 0$$

⑦ Rearrange and make RHS = 0

$$x = \frac{-7 \pm \sqrt{7^2-4(12)(-9)}}{2(12)}$$

$$x = -1.21 \text{ or } 0.622 \text{ (3s.f.)}$$

⑧ Solve for x using quadratic formula or by factorisation.

4) Graph Sketching

Key pieces of information to find before sketching a graph

① Shape ⟶ look at coefficient of x^2 ⟶ If positive : $\cup$ (smiley face ☺)
If negative : $\cap$ (sad face ☹)

② x-intercept(s) ⟶ when $y = 0$

③ y-intercept ⟶ when $x = 0$

④ Turning point coordinates (minimum / maximum) ⟶ From complete the square form

⟶ x-coordinate : $\dfrac{\text{Add the two } x\text{-intercepts}}{2}$ ⟶ y-coordinate : Sub x-coordinate into equation.

optional
⑤ Line of symmetry : $\boxed{x = \quad}$

Forms of quadratic equation		Which pieces of information can be obtained directly
General	$y = ax^2 + bx + c$ $y = 2x^2 + 8x - 7$	✓ Shape ⟶ $\cup$ ✓ y-intercept $= c = -7$
Factorised	$y = \pm(x-h)(x-k)$ $y = (x-3)(x+8)$	✓ Shape ⟶ $\cup$ ✓ x-intercept(s) $= h$ and k $= 3$ and -8
Complete the square	$y = \pm(x-p)^2 + q$ $y = -(x-3)^2 - 42$	✓ Shape ⟶ $\cap$ ✓ Turning point coordinates $= (p, q) = (3, -42)$ (maximum)

Eg 1 : Sketch the graph of $y = (3-x)(x+5)$

when $x = 0$,
$y = (3-0)(0+5)$
$= 15$

max point x-coordinate $= \dfrac{-5+3}{2}$
$= -1$

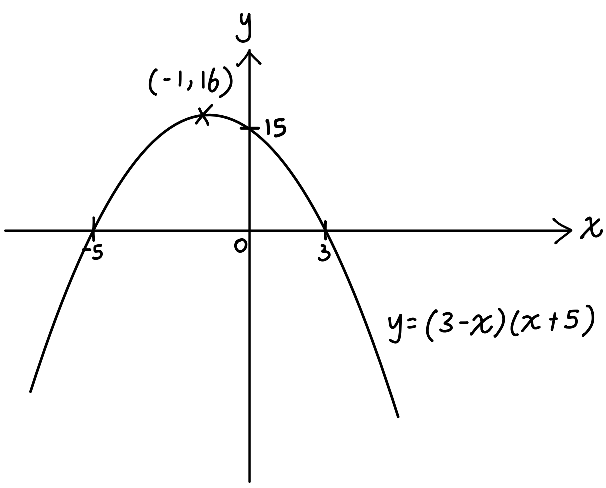
y-coordinate $= 16$

① Shape : $\cap$

② x-intercepts $= -5, 3$

③ y-intercept $= 15$

④ max point $= (-1, 16)$



✱ See next page for Tips on graph sketching !

Eg 2 : Sketch the graph of $y=-x^2+2x-2$ and label the line of symmetry.

when $y=0$,

$-x^2+2x-2=0$

$-(x^2-2x)-2=0$

$-\left[x^2-2x+\left(\frac{-2}{2}\right)^2-\left(\frac{-2}{2}\right)^2\right]-2=0$

$-\left[(x-1)^2-(-1)^2\right]-2=0$

$-(x-1)^2+1-2=0$

$-(x-1)^2-1=0$

$(x-1)^2=-1$

$\therefore$ There is no solution.

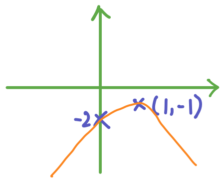
TIP

use calculator : menu, 5,2,2
to check for solution first.
You will see "1+i" and "1−i".
This means there is no x-intercepts.
Complete the square to show no solution
and to find the maximum point coordinates.

① Shape : ∩
② no x-intercept
③ y-intercept = −2
④ max point = (1, −1)

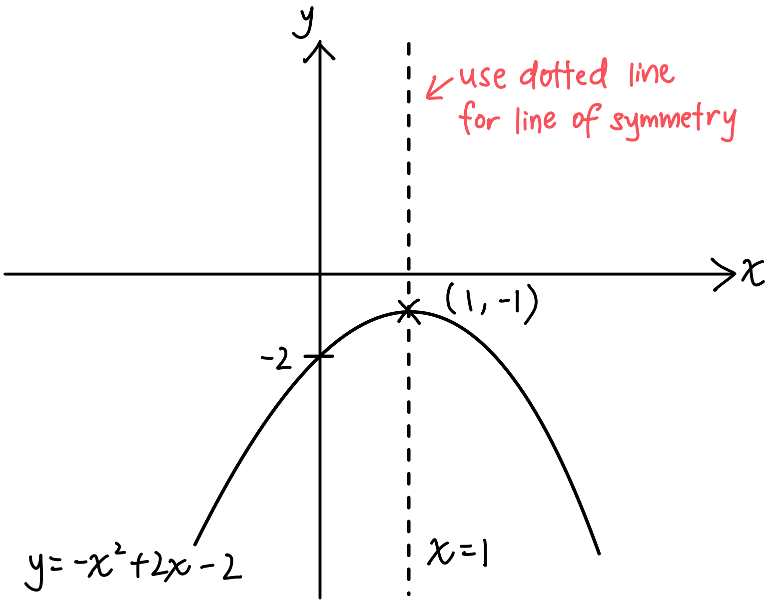
TIP

Sketch a mini graph at the side first to get an idea of where the curve lies.



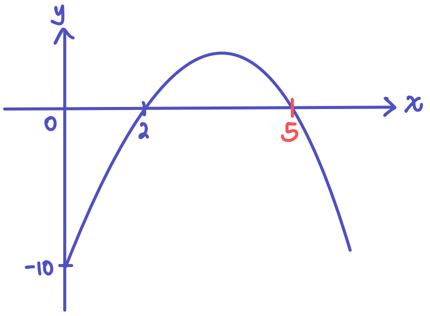
① Mark the points
② connect the points

Then, draw the curve on your actual axes.
Label the intercepts, line of symmetry and equation of the curve.



5) Other common quadratic equations questions

The diagram below shows the curve $y = -x^2 + px - 10$. The curve cuts x-axis at 2 and the y-axis at -10.
[Source: Swiss Cottage Secondary School / 2026 / WA1 / Q4]



(a) Find the value of p.

When $x = 2$, $y = 0$
$0 = -2^2 + 2p - 10$
$0 = 2p - 14$
$2p = 14$
$p = 7$

(b) Find the equation of the line of symmetry.

When $y = 0$,
$-x^2 + 7x - 10 = 0$
$x^2 - 7x + 10 = 0$
$(x-5)(x-2) = 0$
$x - 5 = 0$ or $x - 2 = 0$
$x = 5$ or $x = 2$

	x	-5
x	x^2	$-5x$
-2	$-2x$	10

Line of symmetry: $x = \frac{5+2}{2}$
$x = \frac{7}{2}$

(c) Find the coordinates of the maximum point of the curve.

When $x = \frac{7}{2}$, $y = -\left(\frac{7}{2}\right)^2 + 7\left(\frac{7}{2}\right) - 10$
$y = \frac{9}{4}$
$\therefore$ The coordinates of the maximum point are $\left(\frac{7}{2}, \frac{9}{4}\right)$.

Eg 2: The quadratic equation $ax^2 + bx + c = 0$ has solutions $\frac{4}{5}$ and -2. Given that a, b and c are integers and $a > 0$, find the smallest possible values of a, b and c.

$x = \frac{4}{5}$
$5x = 4$
$5x - 4 = 0$

$x = 2$
$x - 2 = 0$

! Think: why must the question use "smallest"?
If the equation was $2(5x^2 + 6x - 8) = 0$,
we will also get back the solutions $\frac{4}{5}$ and -2.
There are infinite equivalent equations:

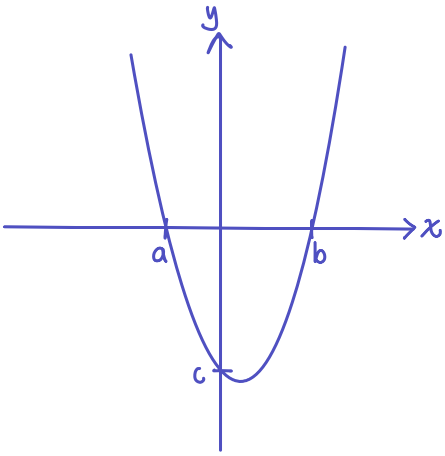
$\left.\begin{array}{l} 6(5x^2 + 6x - 8) = 0 \\ -3(5x^2 + 6x - 8) = 0 \end{array}\right\}$ All give the same solutions.

any number

$(5x-4)(x-2) = 0$
$5x^2 + 10x - 4x - 8 = 0$
$5x^2 + 6x - 8 = 0$

$\therefore a = 5$, $b = 6$ and $c = -8$

The graph of $y = 6x^2 - x - 15$ is shown below. Find the values of a, b and c.



$y = 6x^2 - x - 15$

$y = (3x-5)(2x+3)$

	$3x$	-5
$2x$	$6x^2$	$-10x$
3	$9x$	-15

When $y = 0$,

$(3x-5)(2x+3) = 0$

$3x - 5 = 0$ or $2x + 3 = 0$

$3x = 5$ $\qquad$ $2x = -3$

$x = \frac{5}{3}$ $\qquad$ $x = -\frac{3}{2}$

$\therefore a = -\frac{3}{2}$, $b = \frac{5}{3}$, $c = -15$

The graph of $y = (x+p)^2 + q$ has a turning point at $(-3, -5)$

(a) State the values of p and q.

$p = 3$ and $q = -5$

(b) What is the y-intercept of the graph?

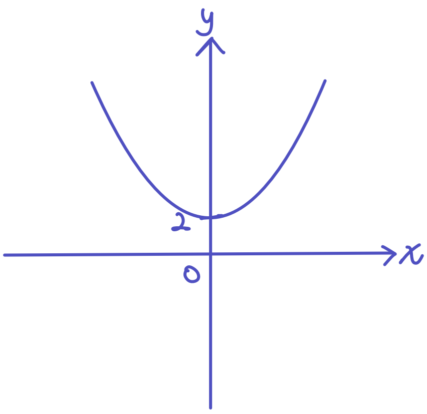
$y = (x+3)^2 - 5$

$y = x^2 + 6x + 9 - 5$

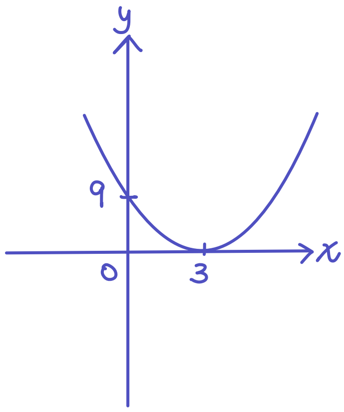
$y = x^2 + 6x + 4$

$\therefore$ y-intercept is 4.

For each of the following graphs, give an example of a quadratic equation in the form of $y = ax^2 + bx + c$, where a, b and c are integers.

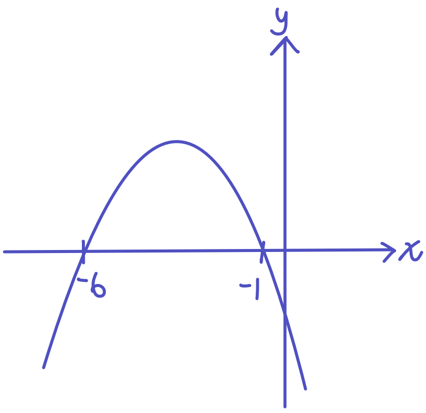


$y = x^2 + 2$



$y = (x-3)^2$

$y = x^2 - 6x + 9$



$y = -(x+6)(x+1)$

$y = -(x^2 + x + 6x + 6)$

$y = -x^2 - 7x - 6$

5) Forming quadratic equations (Real-world context questions)

A railway track covers a distance of 280 km between Two Cities. An Express Train travels faster than a Freight Train. Let the average speed of the Freight Train be x km/h. The average speed of the Express Train is 15 km/h more than the speed of the Freight Train. The difference in the time taken to complete the journey is 30 minutes.

↳ Take note of units

(i) Write down an expression, in terms of x, for the number of hours taken by the Freight Train to complete the journey.

Time taken $= \dfrac{280}{x}$

lower speed → longer time

(ii) Write down an expression, in terms of x, for the number of hours taken by the Express Train to complete the journey.

Time taken $= \dfrac{280}{x+15}$

higher speed → shorter time

(iii) Form an equation to represent this information and show that it reduces to $x^2 + 15x - 8400 = 0$.

$$\frac{280}{x} - \frac{280}{x+15} = \frac{1}{2}$$

$$\frac{280(x+15) - 280x}{x(x+15)} = \frac{1}{2}$$

$$\frac{4200}{x^2 + 15x} = \frac{1}{2}$$

$$8400 = x^2 + 15x$$

$$x^2 + 15x - 8400 = 0 \text{ (shown)}$$

Common mistakes

① Swapping fractions: $\dfrac{280}{x+15} - \dfrac{280}{x}$

② Not changing 30 minutes to $\frac{1}{2}$ hour.

Read question carefully

(iv) Solve the equation $x^2 + 15x - 8400 = 0$, giving your answers correct to 1 decimal place.

$$x = \frac{-15 \pm \sqrt{15^2 - 4(-8400)}}{2}$$

$= 84.458$ or -99.458 (5s.f.) ← Always leave 5s.f. answer to use for next part

$= 84.5$ or -99.5 (1 d.p.) ← Do not reject in this part

(v) Find the time taken by the Freight Train to complete the journey, giving your answer correct to the nearest minute.

$x = 84.458$ or -99.458 (reject as speed > 0)

Time taken $= \dfrac{280}{84.458}$

$= 3.3153\text{h}$ (5s.f.)

$\times 60$

$= 199$ min (nearest minute)