CHAPTER 13 AM Differentiation (Trigo, Expo, Log)

NOTE: Angle x must be in radians.

We can only differentiate sin, cos, tan.

If given other Trigo, use Trigo identities to change!

1) Trigonometric Functions with angles as x.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$y = 5\sin x$

$\frac{dy}{dx} = 5\cos x$

$y = \underbrace{3x^2}_{1\text{ term}}\underbrace{\cos x}_{1\text{ term}}$

$\frac{dy}{dx} = 3x^2(-\sin x) + (\cos x)(6x)$ Product Rule

$= -3x^2\sin x + 6x\cos x$

$y = (4x + 3\tan x)^3$

$\frac{dy}{dx} = 3(4x + 3\tan x)^2(4 + 3\sec^2 x)$ Chain Rule

2) Trigonometric Functions with varying angles and power = 1 (differentiate Trigo × differentiate angle)

$$\frac{d}{dx}[\sin f(x)] = \cos f(x) \times f'(x)$$

$$\frac{d}{dx}[\cos f(x)] = -\sin f(x) \times f'(x)$$

$$\frac{d}{dx}[\tan f(x)] = \sec^2 f(x) \times f'(x)$$

Eg: $f(x) = ax + b$

$\frac{d}{dx}[\sin(ax+b)] = \cos(ax+b) \times a$

$y = 3\tan\left(4x + \frac{\pi}{2}\right)$

Differentiate Trigo — Differentiate angle

$\frac{dy}{dx} = 3\sec^2\left(4x + \frac{\pi}{2}\right) \times 4$

$= 12\sec^2\left(4x + \frac{\pi}{2}\right)$

3) Trigonometric Functions with varying angles and varying powers → Apply chain rule

(Bring down power, power −1 × Differentiate Trigo × Differentiate Angle)

$$\frac{d}{dx}[\sin^n f(x)] = n\sin^{n-1} f(x) \times \cos f(x) \times f'(x)$$

$$\frac{d}{dx}[\cos^n f(x)] = n\cos^{n-1} f(x) \times [-\sin f(x)] \times f'(x)$$

$$\frac{d}{dx}[\tan^n f(x)] = n\tan^{n-1} f(x) \times \sec^2 f(x) \times f'(x)$$

Eg: $f(x) = ax + b$

$\frac{d}{dx}[\cos^3(ax+b)] = 3\cos^2(ax+b) \times [-\sin(ax+b)] \times a$

$y = 4\tan^3\left(\frac{x}{2} + \frac{\pi}{5}\right)$

$= 4\left[\tan\left(\frac{x}{2} + \frac{\pi}{5}\right)\right]^3$

Bring down power, power −1 — Differentiate Trigo — Diff. Angle

$\frac{dy}{dx} = 12\tan^2\left(\frac{x}{2} + \frac{\pi}{5}\right) \times \sec^2\left(\frac{x}{2} + \frac{\pi}{5}\right) \times \frac{1}{2}$

$= 6\tan^2\left(\frac{x}{2} + \frac{\pi}{5}\right)\sec^2\left(\frac{x}{2} + \frac{\pi}{5}\right)$

4) Exponential functions (copy exponential function × differentiate power)

$$\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} \times f'(x)$$

$$\frac{d}{dx}\left(e^{7x+5}\right) = e^{7x+5} \times 7$$
$$= 7e^{7x+5}$$

constant stays

$$\frac{d}{dx}\left(3e^{x^3+4x}\right) = 3e^{x^3+4x} \times (3x^2+4)$$

$$\frac{d}{dx}\left(e^{4x}e^{6x}\right) = \frac{d}{dx}e^{10x} \quad \leftarrow \text{law of indices}$$
$$= 10e^{10x}$$

5) Logarithmic functions $\left(\frac{\text{differentiate function}}{\text{copy function}}\right)$

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln 5x) = \frac{5}{5x} = \frac{1}{x}$$

$$\frac{d}{dx}\left[\ln(3x+2)\right] = \frac{3}{3x+2}$$

Given $y = \ln\sqrt{\frac{2+x}{2-x}}$, find $\frac{dy}{dx}$.

$$y = \ln\left(\frac{2+3x}{2-3x}\right)^{\frac{1}{2}}$$

* Rewrite 'y' by removing root and fraction

$$= \frac{1}{2}\ln\left(\frac{2+3x}{2-3x}\right)$$
$$= \frac{1}{2}\left[\ln(2+3x) - \ln(2-3x)\right]$$

Apply law of logarithm

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{3}{2+3x} - \frac{(-3)}{2-3x}\right)$$
$$= \frac{1}{2}\left[\frac{3(2-3x) + 3(2+3x)}{(2+3x)(2-3x)}\right]$$
$$= \frac{6}{(2+3x)(2-3x)}$$

Differentiate $y = \log_4(3x+2)$

$$y = \frac{\ln(3x+2)}{\ln 4} \longrightarrow \text{constant}$$
$$= \frac{1}{\ln 4}\left[\ln(3x+2)\right]$$
$$\frac{dy}{dx} = \frac{1}{\ln 4}\left(\frac{3}{3x+2}\right)$$
$$= \frac{3}{\ln 4\,(3x+2)}$$

! NOTE:

We can only differentiate ln, not log. If given log, apply change of base formula.

Example questions

The equation of the curve is $y = e^{\sqrt{x}}$. Find the gradient of the curve when $y = e^4$.

$$y = e^{\sqrt{x}}$$
$$y = e^{x^{\frac{1}{2}}}$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$$
$$= \frac{1}{2\sqrt{x}} \times e^{\sqrt{x}}$$
$$= \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

When $y = e^4$,

$$e^{\sqrt{x}} = e^4$$
$$\sqrt{x} = 4$$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$
$$= \frac{e^4}{8}$$
$$= 6.82 \text{ (3s.f.)}$$

The equation of the curve is $y = \ln\frac{3x-1}{3-x}$. Find the range of values of x for which y is an increasing function.

$$y = \ln\frac{3x-1}{3-x}$$
$$y = \ln(3x-1) - \ln(3-x)$$
$$\frac{dy}{dx} = \frac{3}{3x-1} - \frac{(-1)}{3-x}$$
$$= \frac{3}{3x-1} + \frac{1}{3-x}$$
$$= \frac{3(3-x) + 3x-1}{(3x-1)(3-x)}$$
$$= \frac{9-3x+3x-1}{(3x-1)(3-x)}$$
$$= \frac{8}{(3x-1)(3-x)}$$

When y is an increasing function, $\frac{dy}{dx} > 0$

$$\frac{8}{(3x-1)(3-x)} > 0.$$

Since $8 > 0$, for $\frac{dy}{dx} > 0$,

$$(3x-1)(3-x) > 0$$

$$\therefore \frac{1}{3} < x < 3$$

The equation of a curve is $y = \cos 2x - 4\sin x$ for $0 \leq \pi \leq 2\pi$. Find the exact value(s) of x for which the curve has a stationary point.

$$y = \cos 2x - 4\sin x$$
$$\frac{dy}{dx} = -2\sin 2x - 4\cos x$$

At stationary point, $\frac{dy}{dx} = 0$

$$-2\sin 2x - 4\cos x = 0$$
$$\sin 2x + 2\cos x = 0$$
$$2\sin x\cos x + 2\cos x = 0$$
$$\sin x\cos x + \cos x = 0$$
$$\cos x(\sin x + 1) = 0$$

$\cos x = 0$ or $\sin x + 1 = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad$ $\sin x = -1$

$x = \frac{3\pi}{2}$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$