Chapter 11 AM Differentiation Techniques

1) Notation

Question	Differentiate... OR Find the derivative of...	$5x^2$	$y = \frac{1}{2x}$	$f(x) = 3x^3 + 8$
How to present	1^{st} derivative (differentiate once)	$\frac{d}{dx}(5x^2) = 10x$	$\frac{dy}{dx} = -\frac{1}{x^2}$	$f'(x) = 9x^2$
	2^{nd} derivative (differentiate twice)	$\frac{d^2}{dx}(5x^2) = 10$	$\frac{d^2y}{dx^2} = \frac{1}{x^3}$	$f''(x) = 18x$

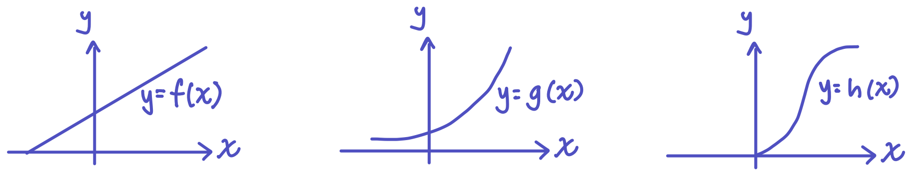
2) What does $\frac{dy}{dx}$ represent?

→ The gradient of the tangent to the graph of $y = f(x)$ at any point

→ The instantaneous rate of change of y with respect to x

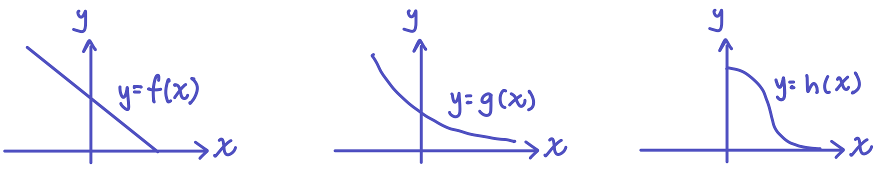
→ if $\frac{dy}{dx} > 0$, then the curve $y = f(x)$ is increasing

Eg :



→ if $\frac{dy}{dx} < 0$, then the curve $y = f(x)$ is decreasing

Eg :



3) What does $\frac{d^2y}{dx^2}$ represent?

→ The rate of change of $\frac{dy}{dx}$ (how the gradient changes)

4) Tips (before differentiating)

✓ Remove roots (indices recap : $\sqrt[3]{x^2} = x^{\frac{2}{3}}$)

✓ Remove fractions (unless using quotient rule)

✓ If 'x' is in the denominator, move it up!

Eg : $\frac{2}{3\sqrt{x}} = \frac{2}{3x^{\frac{1}{2}}} = \frac{2}{3}x^{-\frac{1}{2}}$

5) Algebraic Functions (Bring down power, power − 1)

('a' is a constant)

EXAMPLES

$$\boxed{\frac{d}{dx}(a) = 0}$$

$\frac{d}{dx}(\pi) = 0$ $\frac{d}{dx}\left(-\frac{\sqrt{3}}{2}\right) = 0$

$$\boxed{\frac{d}{dx}(ax^n) = anx^{n-1}}$$

$$\frac{d}{dx}\left(\frac{1}{2}x^{-3}\right) = -\frac{3}{2}x^{-4} = -\frac{3}{2x^4}$$

Multiple terms → differentiate each term independently

$$\boxed{\frac{d}{dx}(ax^n + bx^m + \ldots) = anx^{n-1} + bmx^{m-1} + \ldots}$$

$$y = \frac{4}{9x^3} + 5x\sqrt{x} - \frac{3}{\sqrt[3]{x^2}}$$

$$= \frac{4}{9}x^{-3} + 5x^{\frac{3}{2}} - 3x^{-\frac{2}{3}}$$

$$\frac{dy}{dx} = -\frac{4}{3}x^{-4} + \frac{15}{2}x^{\frac{1}{2}} + 2x^{-\frac{5}{3}}$$

$$= -\frac{4}{3x^4} + \frac{15\sqrt{x}}{2} + \frac{2}{\sqrt[3]{x^5}}$$

6) Chain Rule (Differentiate outside × differentiate inside)

$$\boxed{\frac{d}{dx}\left[(ax+b)^n\right] = n(ax+b)^{n-1}(a)}$$

$$y = \sqrt[3]{2x^2+7} = (2x^2+7)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \underbrace{\frac{1}{3}(2x^2+7)^{-\frac{2}{3}}}_{\text{Differentiate outside}}\underbrace{(4x)}_{\text{Differentiate inside}}$$

$$= \frac{4x}{3\sqrt[3]{(2x^2+7)^2}}$$

7) Product Rule (copy 1st term × diff 2nd term + copy 2nd term × diff 1st term)

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$y = (2x+3)(x-3)^4$$

$$\frac{dy}{dx} = \underbrace{(2x+3)}_{\text{copy 1st term}}\underbrace{\left[4(x-3)^3(1)\right]}_{\text{Differentiate 2nd term using chain rule}} + \underbrace{(x-3)^4}_{\text{copy 2nd term}}\underbrace{(2)}_{\text{Differentiate 1st term}}$$

$$= 4(2x+3)(x-3)^3 + 2(x-3)^4$$

$$= 2(x-3)^3\left[2(2x+3) + (x-3)\right]$$ ← Factorise common term $2(x-3)^3$

$$= 2(x-3)^3(5x+3)$$

8) Quotient Rule $\left(\frac{\text{copy bottom} \times \text{diff. top} - \text{copy top} \times \text{diff. bottom}}{\text{bottom}^2}\right)$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

$$y = \frac{5x+9}{(1-3x^2)}$$

$$\frac{dy}{dx} = \frac{(1-3x^2)(5) - (5x+9)(-6x)}{(1-3x^2)^2}$$

$$= \frac{15x^2 + 54x + 5}{(1-3x^2)^2}$$

9) Increasing and decreasing functions

a) The curve is an increasing / decreasing function

Eg 1: Finding the value(s) / range of unknown

Find the set of values of x for which $f(x) = 2x^3 - 7x^2 + 4x + 3$ is an increasing function.

$f'(x) = 6x^2 - 14x + 4$

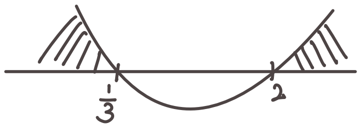
since $f(x)$ is increasing, $f'(x) > 0$

$f'(x) > 0$

$6x^2 - 14x + 4 > 0$

$3x^2 - 7x + 2 > 0$

$(3x-1)(x-2) > 0$



$\therefore x < \frac{1}{3}$ or $x > 2$

Eg 2: Showing / Proving questions

The equation of the curve is $y = -x^3 + 3x^2 - 6x + 5$. Show that the curve is always decreasing for all real values of x.

$\frac{dy}{dx} = -3x^2 + 6x - 6$

$= -3(x^2 - 2x) - 6$ complete the square

$= -3(x-1)^2 - 3$

Since $(x-1)^2 \geq 0$

$-3(x-1)^2 \leq 0$

$-3(x-1)^2 - 3 \leq -3$

$\therefore \frac{dy}{dx} < 0$ and the curve is always decreasing for all real values of x. (shown)

b) The gradient of curve is an increasing or decreasing function.

Eg: The derivative of a function is given by $f'(x) = \frac{3x-1}{x+2}$ for $x > 0$.

Determine if the gradient of the curve is an increasing or decreasing function.

This is asking if the slope is getting steeper. To find the rate of change of the slope, we need to find the second derivative.

$f''(x) = \frac{3(x+2) - (3x-1)}{(x+2)^2}$

$= \frac{3x + 6 - 3x + 1}{(x+2)^2}$

$= \frac{7}{(x+2)^2}$

$f'(x) > 0$: curve is increasing

$f''(x) > 0$: gradient of the curve is increasing.

Since the numerator, 7, and denominator, $(x+2)^2$ are always positive, $f''(x) > 0$.

$\therefore$ The gradient of the curve is an increasing function.

Example questions

Eg 1:

Find the coordinates of the point(s) on the curve $y = 2x^3 + 2x^2 - 5$ where the tangent is perpendicular to the line $2y - 2 - 4x = 0$.

$$y = 2x^3 + 2x^2 - 5$$

$$\frac{dy}{dx} = 6x^2 + 4x$$

Line: $2y - 2 - 4x = 0$

$$2y = 4x + 2$$

$$y = 2x + 1$$

$$(6x^2 + 4x)(2) = -1$$

$$12x^2 + 8x + 1 = 0$$

$$(6x + 1)(2x + 1) = 0$$

$6x + 1 = 0$ or $2x + 1 = 0$

$x = -\frac{1}{6}$ $\qquad$ $x = -\frac{1}{2}$

When $x = -\frac{1}{6}$,

$$y = 2\left(-\frac{1}{6}\right)^3 + 2\left(-\frac{1}{6}\right)^2 - 5$$

$$= -4\frac{103}{108}$$

When $x = -\frac{1}{2}$,

$$y = 2\left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right)^2 - 5$$

$$= -4\frac{3}{4}$$

$\therefore$ The coordinates are $\left(-\frac{1}{6}, -4\frac{103}{108}\right)$ and $\left(-\frac{1}{2}, -4\frac{3}{4}\right)$

Eg 2:

The equation of a curve is $y = \frac{5}{(1-3x)^2} - 12$

(a) Find the first and second derivatives.

$$\frac{dy}{dx} = \frac{(1-3x)^2(0) - 5\left[2(1-3x)(-3)\right]}{(1-3x)^4}$$

$$= \frac{-5\left[-6(1-3x)\right]}{(1-3x)^4}$$

$$= \frac{30(1-3x)}{(1-3x)^4}$$

$$= \frac{30}{(1-3x)^3}$$

$$\frac{d^2y}{dx^2} = \frac{(1-3x)^3(0) - 30 \times 3(1-3x)^2(-3)}{(1-3x)^6}$$

$$= \frac{270(1-3x)^2}{(1-3x)^6}$$

$$= \frac{270}{(1-3x)^4}$$

(b) Find the gradient of the curve at the y-axis.

At y-axis, $x = 0$

$$\frac{dy}{dx} = \frac{30}{(1-3\times 0)^3}$$

$$= 30$$