Chapter 1 AM Quadratic Functions

1) Complete the square (Changing $y = ax^2 + bx + c$ to $y = a(x-h)^2 + K$)

$$x^2 + bx + c = \underbrace{x^2 + bx + \left(\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right)^2 + c$$
$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

"number in front of x^2"

* coefficient of x^2 must be positive 1 before you start completing the square.

Example 1 : Express the following in the form of $a(x-h)^2 + K$.

	Steps	Example : $x^2 + 10x$ $\boxed{b = 10}$
1)	Copy the x^2 term and bx term. Add $\left(\frac{b}{2}\right)^2$, subtract $\left(\frac{b}{2}\right)^2$. Copy the constant.	$= \underbrace{x^2 + 10x + \left(\frac{10}{2}\right)^2} - \left(\frac{10}{2}\right)^2$
2)	First three terms becomes $\left(x + \frac{b}{2}\right)^2$	$= \left(x + \frac{10}{2}\right)^2 - \underbrace{\left(\frac{10}{2}\right)^2}$
3)	Combine the constants.	$= (x+5)^2 - 25$

Example 2 : When coefficient of x^2 is NOT $+1$.

	Steps	Example : $-3x^2 + 5x + 7$
1)	Factorise the coefficient of x^2. Leave the constant alone.	$-3x^2 + 5x + 7$ $= -3\left(x^2 - \frac{5}{3}x\right) + 7$ $\boxed{b = -\frac{5}{3}}$ complete the square for this bracket. Everything else outside the bracket stays the same.
2)	Copy the x^2 term and bx term. Add $\left(\frac{b}{2}\right)^2$, subtract $\left(\frac{b}{2}\right)^2$. Copy the constant.	$= -3\left[\underbrace{x^2 - \frac{5}{3}x + \left(\frac{-\frac{5}{3}}{2}\right)^2} - \left(\frac{-\frac{5}{3}}{2}\right)^2\right] + 7$
3)	First three terms becomes $\left(x + \frac{b}{2}\right)^2$	$= -3\left[\left(x + \frac{-\frac{5}{3}}{2}\right)^2 - \left(\frac{-\frac{5}{3}}{2}\right)^2\right] + 7$
4)	Expand the outer bracket.	$= -3\left(x + \frac{-\frac{5}{3}}{2}\right)^2 \underbrace{+ 3\left(\frac{-\frac{5}{3}}{2}\right)^2 + 7}$
5)	Combine the constants.	$= -3\left(x - \frac{5}{6}\right)^2 + \frac{109}{12}$

2) Graph Sketching

Key pieces of information to find before sketching a graph

1. Shape ⟶ look at coefficient of x^2 ⟶ If positive : $\cup$ (smiley face)
   If negative : $\cap$ (sad face)
2. x-intercept(s) ⟶ when $y = 0$
3. y-intercept ⟶ when $x = 0$
4. Turning point coordinates (minimum / maximum) ⟶ From complete the square form : (h, k)
   ⟶ x-coordinate : $\dfrac{\text{Add the two } x\text{-intercepts}}{2}$ ⟶ y-coordinate : Sub x-coordinate into equation.

optional

5. Line of symmetry : $\boxed{x = \quad}$

Forms of quadratic equation		Which pieces of information can be obtained directly
General	$y = ax^2 + bx + c$ $y = 9x^2 - 8x + 13$	✓ Shape ⟶ $\cup$ ✓ y-intercept = c = 13
Factorised	$y = \pm a(x-p)(x-q)$ $y = 2(x+3)(x-8)$	✓ Shape ⟶ $\cup$ ✓ x-intercept = (p, q) = -3 or 8
Complete the square	$y = \pm a(x-h)^2 + k$ $y = -5(x+7)^2 + 15$	✓ Shape ⟶ $\cap$ ✓ Turning point coordinates = (h, k) = $(-7, 15)$ (maximum)

Eg 1: (a) Express $4x^2-16x+3$ in the form of $a(x-h)^2+k$ and state the coordinates of the turning point of the graph of $f(x) = 4x^2-16x+3$

(b) Hence, sketch the graph and state its maximum or minimum value.

(a)
$$
\begin{aligned}
4x^2-16x+3 &= 4(x^2-4x)+3 \\
&= 4\left[x^2-4x+\left(\tfrac{-4}{2}\right)^2-\left(\tfrac{-4}{2}\right)^2\right]+3 \\
&= 4\left[(x-2)^2-4\right]+3 \\
&= 4(x-2)^2-16+3 \\
&= 4(x-2)^2-13
\end{aligned}
$$

The coordinates of the turning point is $(2,-13)$.

(b)

① Shape : ∪

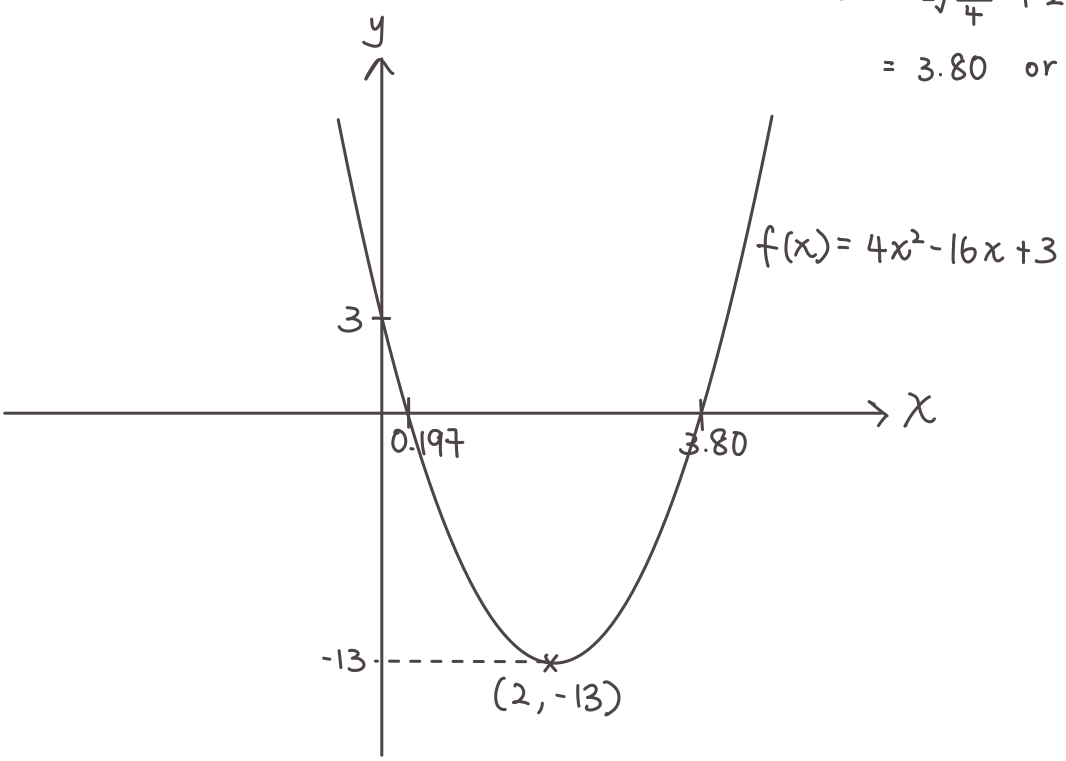
② x-intercepts = 0.197, 3.80

③ y-intercept = 3

④ max point = $(2,-13)$

when $f(x)=0$,

$$
\begin{aligned}
4x^2-16x+3 &= 0 \\
4(x-2)^2-13 &= 0 \\
4(x-2)^2 &= 13 \\
(x-2)^2 &= \frac{13}{4} \\
x-2 &= \pm\sqrt{\frac{13}{4}} \\
x &= \pm\sqrt{\frac{13}{4}}+2 \\
&= 3.80 \text{ or } 0.197 \text{ (3s.f.)}
\end{aligned}
$$



The minimum value is -13.

3) Understanding the Question

$$y = a(x-h)^2 + k$$

When the question asks for...

- Maximum or minimum value ⟶ y-value of the turning point $= k$
- Turning point coordinates ⟶ write as (h, k).

4) Whether curve lies completely above / below x-axis / a particular value

① Complete the square.

② State conditions and conclude:

Since the coefficient of x^2 = ☐ which is ☐ >0 ☐ <0, and the ☐ minimum ☐ maximum value of the curve is ☐ positive ☐ negative, then the curve lies completely ☐ below ☐ above the x-axis.

Eg: Explain why the value of $y = 2x^2 + 3x - 2$ will never be less than -4.

$$\begin{aligned}
y &= 2x^2 + 3x - 2 \\
&= 2\left(x^2 + \tfrac{3}{2}x\right)^2 - 2 \\
&= 2\left[x^2 + \tfrac{3}{2}x + \left(\tfrac{\frac{3}{2}}{2}\right)^2 - \left(\tfrac{\frac{3}{2}}{2}\right)^2\right] - 2 \\
&= 2\left[\left(x + \tfrac{3}{4}\right)^2 - \tfrac{9}{16}\right] - 2 \\
&= 2\left(x + \tfrac{3}{4}\right)^2 - \tfrac{9}{8} - 2 \\
&= 2\left(x + \tfrac{3}{4}\right)^2 - \tfrac{25}{8}
\end{aligned}$$

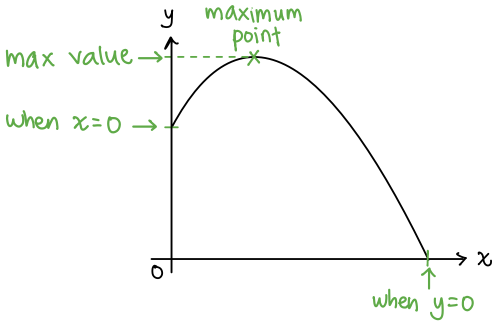
Since the coefficient of $x^2 = 2$ which is >0, and the minimum value of the curve is $-3\frac{1}{8}$ which is > -4, then the curve lies completely above $y = -4$ and the value of $y = 2x^2 + 3x - 2$ will never be less than -4.

Chap 1 & 2 Guideline

- Maximum / Minimum value or coordinate
  ⟶ Complete the square
- Whether the curve lies completely above or below x-axis / curve is always positive or negative
  ⟶ Use Completing the Square or Discriminant (Chap 2)
- Number of roots / intersections
  ⟶ Discriminant (Chap 2)

5) Application Questions

TIP Sketch the curve (if not given) to help visualise the problem.



Eg: N2023/P1/Q12

A ball is thrown vertically upwards. Its height, h m, above the ground at time t seconds after being thrown is given by the formula $h = 1.75 + 5t - 5t^2$.

(a) State the height above the ground from which the ball is thrown.

When $t = 0$ s, $h = 1.75$

(b) Express h in the form of $a + b(t+c)^2$ where a, b and c are constants to be determined.

$$\begin{aligned} h &= 1.75 + 5t - 5t^2 \\ &= -5t^2 + 5t + 1.75 \\ &= -5(t^2 - t) + 1.75 \\ &= -5\left[\left(t - \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2\right] + 1.75 \\ &= 3 - 5(t - 0.5)^2 \end{aligned}$$

$\therefore a = 3$, $b = -5$ and $c = -0.5$

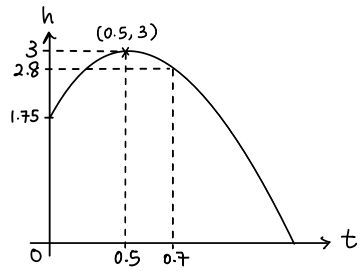
(c) Hence, state the maximum height attained by the ball and the time at which this occurs.

Maximum height = 3 m when $t = 0.5$ s.

extended:

(d) Find the height of the ball 0.7 seconds after it has been thrown, stating whether it is moving upwards or downwards.

When $t = 0.7$, $h = 1.75 + 5(0.7) - 5(0.7)^2$
$= 2.8$



The height of the ball 0.7s after it has been thrown is 2.8m.

The maximum height is reached at $t = 0.5$s. Since $0.7 > 0.5$, the ball is moving downwards.

extended:

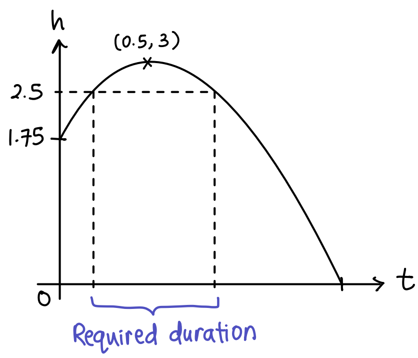
(e) Find the duration for which the ball is at least 2.5m above the ground.

When $h = 2.5$, $1.75 + 5t - 5t^2 = 2.5$

$$-5t^2 + 5t - 0.75 = 0$$

$$t = \frac{-(5) \pm \sqrt{5^2 - 4(-5)(-0.75)}}{2(-5)}$$

$= 0.18377$ or 0.81622 (5s.f.)



Duration = $0.81622 - 0.18377$
$= 0.632$ s (3s.f.)