

CHAPTER
13

AM

Differentiation
(Trigo, Expo, Log)

! NOTE:

Angle x must be in radians.We can only differentiate \sin, \cos, \tan .
If given other Trigo, use Trigo identities to change!1) Trigonometric Functions with angles as x .

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$y = 5 \sin x$$

$$\frac{dy}{dx} = 5 \cos x$$

$$y = \overbrace{3x^2}^{1 \text{ term}} \overbrace{\cos x}^{1 \text{ term}}$$

$$\frac{dy}{dx} = 3x^2(-\sin x) + (\cos x)(6x) \quad \text{Product Rule}$$

$$= -3x^2 \sin x + 6x \cos x$$

$$y = (4x + 3 \tan x)^3$$

$$\frac{dy}{dx} = 3(4x + 3 \tan x)^2 (4 + 3 \sec^2 x) \quad \text{Chain Rule}$$

2) Trigonometric Functions with varying angles (differentiate Trigo \times differentiate angle) and power = 1

$$\frac{d}{dx}[\sin f(x)] = \cos f(x) \times f'(x)$$

$$\frac{d}{dx}[\cos f(x)] = -\sin f(x) \times f'(x)$$

$$\frac{d}{dx}[\tan f(x)] = \sec^2 f(x) \times f'(x)$$

Eg: $f(x) = ax + b$

$$\frac{d}{dx}[\sin(ax+b)] = \cos(ax+b) \times a$$

$$y = 3 \tan\left(4x + \frac{\pi}{2}\right)$$

$$\frac{dy}{dx} = \overbrace{3 \sec^2\left(4x + \frac{\pi}{2}\right)}^{\text{Differentiate Trigo}} \times \overbrace{4}^{\text{Differentiate angle}}$$

$$= 12 \sec^2\left(4x + \frac{\pi}{2}\right)$$

3) Trigonometric Functions with varying angles and varying powers \rightarrow Apply chain rule
(Bring down power, power - 1 \times Differentiate Trigo \times Differentiate Angle)

$$\frac{d}{dx}[\sin^n f(x)] = n \sin^{n-1} f(x) \times \cos f(x) \times f'(x)$$

$$\frac{d}{dx}[\cos^n f(x)] = n \cos^{n-1} f(x) \times [-\sin f(x)] \times f'(x)$$

$$\frac{d}{dx}[\tan^n f(x)] = n \tan^{n-1} f(x) \times \sec^2 f(x) \times f'(x)$$

Eg: $f(x) = ax + b$

$$\frac{d}{dx}[\cos^3(ax+b)] = 3 \cos^2(ax+b) \times [-\sin(ax+b)] \times a$$

$$y = 4 \tan^3\left(\frac{x}{2} + \frac{\pi}{5}\right)$$

$$= 4 \left[\tan\left(\frac{x}{2} + \frac{\pi}{5}\right)\right]^3$$

$$\frac{dy}{dx} = \overbrace{12 \tan^2\left(\frac{x}{2} + \frac{\pi}{5}\right)}^{\text{Bring down power, power - 1}} \times \overbrace{\sec^2\left(\frac{x}{2} + \frac{\pi}{5}\right)}^{\text{Differentiate Trigo}} \times \overbrace{\frac{1}{2}}^{\text{Diff. Angle}}$$

$$= 6 \tan^2\left(\frac{x}{2} + \frac{\pi}{5}\right) \sec^2\left(\frac{x}{2} + \frac{\pi}{5}\right)$$

4) Exponential functions (copy exponential function \times differentiate power)

$$\frac{d}{dx} [e^{f(x)}] = e^{f(x)} \times f'(x)$$

$$\frac{d}{dx} (e^{7x+5}) = e^{7x+5} \times 7$$

$$= 7e^{7x+5}$$

$$\frac{d}{dx} (3e^{x^3+4x}) = 3e^{x^3+4x} \times (3x^2+4)$$

constant stays
↓

$$\frac{d}{dx} (e^{4x} e^{6x}) = \frac{d}{dx} e^{10x} \quad \leftarrow \text{law of indices}$$

$$= 10e^{10x}$$

5) Logarithmic functions ($\frac{\text{differentiate function}}{\text{copy function}}$)

$$\frac{d}{dx} [\ln f(x)] = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

$$\frac{d}{dx} (\ln 5x) = \frac{5}{5x} = \frac{1}{x}$$

$$\frac{d}{dx} [\ln(3x+2)] = \frac{3}{3x+2}$$

Given $y = \ln \sqrt{\frac{2+x}{2-x}}$, find $\frac{dy}{dx}$.

$$y = \ln \left(\frac{2+3x}{2-3x} \right)^{\frac{1}{2}} \quad \text{* Rewrite 'y' by removing root and fraction}$$

$$= \frac{1}{2} \ln \left(\frac{2+3x}{2-3x} \right) \quad \text{] Apply law of logarithm}$$

$$= \frac{1}{2} [\ln(2+3x) - \ln(2-3x)]$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{3}{2+3x} - \frac{(-3)}{2-3x} \right)$$

$$= \frac{1}{2} \left[\frac{3(2-3x) + 3(2+3x)}{(2+3x)(2-3x)} \right]$$

$$= \frac{6}{(2+3x)(2-3x)}$$

! NOTE:

We can only differentiate \ln , not \log . If given \log , apply change of base formula.

Differentiate $y = \log_4(3x+2)$

$$y = \frac{\ln(3x+2)}{\ln 4} \rightarrow \text{constant}$$

$$= \frac{1}{\ln 4} [\ln(3x+2)]$$

$$\frac{dy}{dx} = \frac{1}{\ln 4} \left(\frac{3}{3x+2} \right)$$

$$= \frac{3}{\ln 4(3x+2)}$$

Example questions

The equation of the curve is $y = e^{\sqrt{x}}$.
Find the gradient of the curve when $y = e^4$.

$$y = e^{\sqrt{x}}$$

$$y = e^{x^{\frac{1}{2}}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} x^{-\frac{1}{2}} e^{x^{\frac{1}{2}}} \\ &= \frac{1}{2\sqrt{x}} \times e^{\sqrt{x}} \\ &= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \end{aligned}$$

When $y = e^4$,

$$e^{\sqrt{x}} = e^4$$

$$\sqrt{x} = 4$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \\ &= \frac{e^4}{8} \\ &= 6.82 \text{ (3s.f.)} \end{aligned}$$

The equation of the curve is $y = \ln \frac{3x-1}{3-x}$. Find the range of values of x for which y is an increasing function.

$$y = \ln \frac{3x-1}{3-x}$$

$$y = \ln(3x-1) - \ln(3-x)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{3x-1} - \frac{(-1)}{3-x} \\ &= \frac{3}{3x-1} + \frac{1}{3-x} \\ &= \frac{3(3-x) + 3x-1}{(3x-1)(3-x)} \\ &= \frac{9-3x+3x-1}{(3x-1)(3-x)} \\ &= \frac{8}{(3x-1)(3-x)} \end{aligned}$$

When y is an increasing function, $\frac{dy}{dx} > 0$

$$\frac{8}{(3x-1)(3-x)} > 0.$$

Since $8 > 0$, for $\frac{dy}{dx} > 0$,

$$(3x-1)(3-x) > 0$$



$$\therefore \frac{1}{3} < x < 3$$

The equation of a curve is $y = \sin 2x - 4\cos x$ for $0 \leq x \leq 2\pi$.

Find the exact value(s) of x for which the curve has a stationary point.

$$y = \cos 2x - 4\sin x$$

$$\frac{dy}{dx} = -2\sin 2x - 4\cos x$$

At stationary point, $\frac{dy}{dx} = 0$

$$-2\sin 2x - 4\cos x = 0$$

$$\sin 2x + 2\cos x = 0$$

$$2\sin x \cos x + 2\cos x = 0$$

$$\sin x \cos x + \cos x = 0$$

$$\cos x (\sin x + 1) = 0$$

$$\cos x = 0 \quad \text{or} \quad \sin x + 1 = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \sin x = -1$$

$$x = \frac{3\pi}{2}$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$